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INTERPOLATION ON COUNTABLY MANY ALGEBRAIC SUBSETS FOR WEIGHTED ENTIRE FUNCTIONS

SHIGEKI OH'UCHI

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1. Introduction

Let X_ν ($\nu \in \mathbb{N}$, the set of positive integers) be k_ν-codimensional complex affine subspaces of $\mathbb{C}^n$ ($1 \leq k_\nu \leq n$). Assume that $X_\nu \cap X_{\nu'} = \emptyset$ for $\nu \neq \nu'$. Let N_ν be the orthogonal complement of X_ν, where we use the canonical inner product $\langle z, w\rangle = \sum_{l=1}^n z_l \bar{w}_l$ on $\mathbb{C}^n$. Set $S_\nu = N_\nu \cap S^{2n-1}$, where $S^{2n-1} = \{u \in \mathbb{C}^n : |u| = 1\}$. Then Oh'uchi [10] proved the following result:

Theorem A. *Let $X = \bigcup_{\nu\in\mathbb{N}} X_\nu$ be an analytic subset of $\mathbb{C}^n$ consisting of disjoint complex affine subspaces X_ν. Let p be a weight function on $\mathbb{C}^n$. Then X is interpolating for $A_p(\mathbb{C}^n)$ if and only if there exist $f_1, \ldots, f_m \in A_p(\mathbb{C}^n)$ ($m \geq \sup_{\nu\in\mathbb{N}} k_\nu$) and constants ε, $C > 0$ such that*

$$X \subset Z(f_1, \ldots, f_m) = \{z \in \mathbb{C}^n : f_1(z) = \cdots = f_m(z) = 0\} \tag{1.1}$$

and

$$\sum_{j=1}^m |D_u f_j(\zeta)| \geq \varepsilon \exp(-Cp(\zeta)) \tag{1.2}$$

for all $u \in S_\nu$, $\zeta \in X_\nu$ and $\nu \in \mathbb{N}$.

Here the directional derivative $D_u f$ with a vector $u = (u_1, \ldots u_n) \in S^{2n-1}$ is defined by

$$D_u f = \sum_{l=1}^n \frac{\partial f}{\partial z_l} \cdot u_l.$$

Note that by the proof of Theorem A in [10] the above m may be set equal to $\sup_{\nu\in\mathbb{N}} k_\nu$ when X is interpolating for $A_p(\mathbb{C}^n)$. For the terminologies, see §2. It ex-

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tends the result of Berenstein and Li [2, Theorem 2.5], which deals with the case of $k_\nu = n$ for all $\nu \in \mathbb{N}$.

In the present paper, we would like to discuss the case where X_ν are algebraic subsets, not necessarily affine linear. Because of the difficulties to deal with in general, we formulate this problem as follows. It is first noted that Theorem A implies the following corollary:

Corollary 1.1. *Let $p_m(z_1, \dots, z_m) = q(|z|)$ be a radial weight function on $\mathbb{C}^m$ and set $p_n(z_1, \dots, z_n) = q(|z|)$, which is a radial weight function on $\mathbb{C}^n$ $(m < n)$. Let $X = \{\zeta_\nu\}_{\nu\in\mathbb{N}}$ be a discrete variety in $\mathbb{C}^m$. Then $X \times \mathbb{C}^{n-m}$ is interpolating for $A_{p_n}(\mathbb{C}^n)$ if and only if X is interpolating for $A_{p_m}(\mathbb{C}^m)$.*

Corollary 1.1 can be restated as follows: Define a mapping $F = (F_1, \dots, F_m) : \mathbb{C}^n \to \mathbb{C}^m$ by $F_j(z) = z_j$ $(j = 1, \dots, m)$. Then $F^{-1}(X)$ is interpolating for $A_p(\mathbb{C}^n)$ if and only if X is interpolating for $A_p(\mathbb{C}^m)$. Conversely, when F is a linear mapping from $\mathbb{C}^n$ onto $\mathbb{C}^m$ with $\operatorname{rank} F = m$, we can reduce the interpolation problem for $F^{-1}(X)$ to that for $X' \times \mathbb{C}^{n-m}$, where X' is the image of X by some linear mapping determined by F and X. By [2, Theorem 2.5], X' is interpolating for $A_p(\mathbb{C}^m)$ if and only if X is interpolating for $A_p(\mathbb{C}^n)$. The main result of this paper is as follows:

Main Theorem. *Suppose that $m \leq n$. Let $X = \{\zeta_\nu\}_{\nu\in\mathbb{N}}$ be a discrete variety in $\mathbb{C}^m$ and let $F = (F_1, \dots, F_m) \in \mathbb{C}[z_1, \dots, z_n]^m$. Put $d = \max_{j=1,\dots,m} \deg F_j$. For $a > 0$, we assume that*

(1) *X is interpolating for $A_{|\,\cdot\,|^a}(\mathbb{C}^m)$;*

(2) *there exist constants ε, $C > 0$ and a finite subset E of $\mathbb{N}$ such that*

$$\sum_{\kappa=1}^{\binom{n}{m}} |\triangle_\kappa^F(z)| \geq \varepsilon \exp(-C|z|^{ad})$$

for all $z \in F^{-1}(\zeta_\nu)$, $\nu \in \mathbb{N} \setminus E$.

Here the sum is taken over all $m \times m$ minors $\triangle_\kappa^F$ of Jacobian matrix JF. Then $F^{-1}(X)$ is interpolating for $A_{|\,\cdot\,|^b}(\mathbb{C}^n)$ for every $b \geq ad$.

REMARK. If $F : \mathbb{C}^n \to \mathbb{C}^m$ is the standard projection with $\operatorname{rank} F = m$ and $p(z) = |z|^a$, then the sufficiency part of Corollary 1.1 is deduced from the main theorem, where $d = 1$ and $b = ad = a$.

2. Preliminaries

We fix the notation. A plurisubharmonic function $p : \mathbb{C}^n \to [0, \infty)$ is called a *weight function* if it satisfies

$$\log(1+|z|^2) = O(p(z)) \tag{2.1}$$

and there exist constants C_1, $C_2 > 0$ such that for all z, z' with $|z - z'| \leq 1$

$$p(z') \leq C_1 p(z) + C_2. \tag{2.2}$$

A weight function p is said to be *radial* if

$$p(z) = p(|z|). \tag{2.3}$$

DEFINITION 2.1. Let $\mathcal{O}(\mathbb{C}^n)$ be the ring of all entire functions on $\mathbb{C}^n$ and let p be a weight function on $\mathbb{C}^n$. Set

$$A_p(\mathbb{C}^n) = \{f \in \mathcal{O}(\mathbb{C}^n) : \text{There exist constants } A, B > 0 \text{ such that } |f(z)| \leq A\exp(Bp(z)) \text{ for all } z \in \mathbb{C}^n.\}.$$

Then $A_p(\mathbb{C}^n)$ is a subring of $\mathcal{O}(\mathbb{C}^n)$. The following lemma is easily deduced from (2.1) and (2.2):

Lemma 2.2. *Let p be a weight function on $\mathbb{C}^n$. Then the following hold*:
(1) $\mathbb{C}[z_1, \ldots, z_n] \subset A_p(\mathbb{C}^n)$.
(2) *If $f \in A_p(\mathbb{C}^n)$, then $\partial f/\partial z_j \in A_p(\mathbb{C}^n)$ for $j = 1, \ldots, n$.*
(3) *$f \in \mathcal{O}(\mathbb{C}^n)$ belongs to $A_p(\mathbb{C}^n)$ if and only if there exists a constant $K > 0$ such that*

$$\int_{\mathbb{C}^n} |f|^2 \exp(-Kp)d\lambda < \infty,$$

where $d\lambda$ denotes the Lebesgue measure on $\mathbb{C}^n$.

For the proof, see e.g. [8].

EXAMPLE 2.3. (1) If $p(z) = \log(1+|z|^2)$, then $A_p(\mathbb{C}^n) = \mathbb{C}[z_1, \ldots, z_n]$.
(2) If $p(z) = |z|^a$ $(a > 0)$, then $A_p(\mathbb{C}^n)$ is the space of entire functions which are of order $= a$ and of finite type, or which are of order $< a$.
(3) If $p(z) = |\operatorname{Im} z| + \log(1+|z|^2)$, then $A_p(\mathbb{C}^n) = \hat{\mathcal{E}}'(\mathbb{R}^n)$, that is, the space of Fourier transforms of distributions with compact support on $\mathbb{R}^n$ (see e.g. [7]).
(4) When $p(z) = \exp|z|^a$ $(a > 0)$, p is a weight function if and only if $a \leq 1$.

In the rest of this paper, p will always represent a weight function.

DEFINITION 2.4. Let X be an analytic subset of $\mathbb{C}^n$, and let $\mathcal{O}(X)$ be the space of analytic functions on X. Then we define

$$A_p(X) = \{f \in \mathcal{O}(X) : \text{There exist constants } A, B > 0 \text{ such that } |f(z)| \le A\exp(Bp(z)) \text{ for all } z \in X.\}.$$

DEFINITION 2.5. An analytic subset X of $\mathbb{C}^n$ is said to be *interpolating for* $A_p(\mathbb{C}^n)$ if the restriction map $R_X : A_p(\mathbb{C}^n) \to A_p(X)$ defined by $R_X(f) = f|_X$ is surjective.

The semilocal interpolation theorem by [4] is useful to show an analytic subset to be interpolating. Let X be given by

$$X = Z(f_1, \dots, f_N) = \{z \in \mathbb{C}^n : f_1(z) = \cdots = f_N(z) = 0\}$$

with $f_1, \dots, f_N \in A_p(\mathbb{C}^n)$. Then for ε, $C > 0$, we define

$$S_p(f; \varepsilon, C) = \left\{ z \in \mathbb{C}^n : |f(z)| = \left(\sum_{j=1}^{N} |f_j(z)|^2 \right)^{1/2} < \varepsilon \exp(-Cp(z)) \right\},$$

which is an open neighborhood of X. We recall the semilocal interpolation theorem of [4].

Semilocal Interpolation Theorem. *Let h be a holomorphic function in $S_p(f; \varepsilon, C)$ such that*

$$|h(z)| \le A_1 \exp(B_1 p(z))$$

for all $z \in S_p(f; \varepsilon, C)$, where ε, $C > 0$. Then there exist an entire function $H \in A_p(\mathbb{C}^n)$, constants ε_1, C_1, A, $B > 0$ and holomorphic functions $g_1, \dots, g_N$ in $S_p(f; \varepsilon_1, C_1)$ such that

$$H(z) - h(z) = \sum_{j=1}^{N} g_j(z) f_j(z)$$

and

$$|g_j(z)| \le A \exp(Bp(z))$$

for all $z \in S_p(f; \varepsilon_1, C_1)$ and $j = 1, \dots, N$. In particular, $H = h$ on the variety $X = Z(f_1, \dots, f_N)$.

3. A_p-interpolation on algebraic subsets

To prove the main theorem, we first show the following result:

Theorem 3.1. *Every algebraic subset $V \subset \mathbb{C}^n$ is interpolating for $A_p(\mathbb{C}^n)$.*

We assume that V is irreducible until we begin the proof of Theorem 3.1 after Lemma 3.17. Then we have the prime ideal $I_V \subset \mathbb{C}[z_1, \dots, z_n]$ such that $V = I_V^{-1}(0) = \{z \in \mathbb{C}^n : P(z) = 0 \text{ for all } P \in I_V\}$. Defining the terminology, we state the normalization theorem.

Normalization Theorem. *After a suitable linear change of coordinates, the following conditions hold*:
(1) *There exists $k \in \{0, 1, \dots, n-1\}$ such that $I_V \cap \mathbb{C}[z_1, \dots, z_k] = \{0\}$ and the factor ring $\mathbb{C}[z_1, \dots, z_n]/I_V$ is a finitely generated $\mathbb{C}[z_1, \dots, z_k]$-module.*
Here we set $z' = (z_1, \dots, z_k) \in \mathbb{C}^k$ and $z'' = (z_{k+1}, \dots, z_n) \in \mathbb{C}^{n-k}$.
(2) *There exists $C_0 > 0$ such that $|z_{k+j}| \le C_0(1+|z'|)$ for all $z \in V$ and $j = 1, \dots, n-k$.*
(3) *I_V contains irreducible polynomials*

$$Q_j(z', z_{k+j}) = z_{k+j}^{\mu} + q_{j,1}(z')z_{k+j}^{\mu-1} + \cdots + q_{j,\mu}(z')$$

of degree μ, where $q_{j,\nu} \in \mathbb{C}[z_1, \dots, z_k]$.
Let $\alpha_1(z'), \dots, \alpha_\mu(z')$ be the roots of $Q_1(z', z_{k+1})$ as a polynomial in z_{k+1}. Then we denote by $\Delta(z')$ the discriminant of Q_1 as a polynomial in z_{k+1}, that is,

$$\Delta(z') = \prod_{\nu \ne \nu'} (\alpha_\nu(z') - \alpha_{\nu'}(z')).$$

(4) *We have polynomials $T_j \in \mathbb{C}[z_1, \dots, z_k, z_{k+j}]$ $(j = 2, \dots, n-k)$ with $\Delta(z')z_{k+j} - T_j(z', z_{k+j}) \in I_V$.*
Put $V_0 = V \setminus \Delta^{-1}(0)$.
(5) *V_0 is an open dense subset of V and a μ-dimensional complex submanifold of $\mathbb{C}^n \setminus \Delta^{-1}(0)$.*
Let $\pi_V : \mathbb{C}^n \ni z = (z', z'') \mapsto z' \in \mathbb{C}^k$ be the projection.
(6) *π_V is a finite μ-fold covering map from V_0 onto $\mathbb{C}^k \setminus \Delta^{-1}(0)$.*

For the proof, see e.g. [6, Theorem A.1.1 in Chapter 3], [9, Proposition 7.7.3].

For $\varepsilon > 0$, $N > 0$ and $\xi \in \mathbb{C}^n$, we define the polydisc

$$D_{\varepsilon,N}(\xi) = \{z \in \mathbb{C}^n : |z_j - \xi_j| < \varepsilon(1 + |\xi|)^{-N} \ (\forall j = 1, \dots, n)\}.$$

For the given $f \in A_p(V)$, we take $A,\ B > 0$ such that

$$|f(z)| \leq A \exp(Bp(z)), \quad \forall z \in V.$$

Lemma 3.2. *We have $\varepsilon,\ N,\ A_1,\ B_1 > 0$ satisfying: for all $\xi \in V$ there exists $F \in \mathcal{O}(D_{\varepsilon,N}(\xi))$ such that $\Delta f - F = 0$ on $V \cap D_{\varepsilon,N}$ and*

$$|F(z)| \leq A_1 \exp(B_1 p(z)), \quad z \in D_{\varepsilon,N}(\xi).$$

Proof. If $\dim V = k = 0$, V consists of only one point, so the lemma is trivial. Then we assume that $1 \leq k \leq n-1$. To apply the normalization theorem, we give a suitable linear change of coordinates. Set

$$D'_{\varepsilon,N}(\xi) = \{z' \in \mathbb{C}^k : |z_j - \xi_j| < \varepsilon(1+|\xi|)^{-N} \ (\forall j = 1, \ldots, k)\}.$$

Here we need the following lemma:

Lemma 3.3. *There exists $\varepsilon > 0$ such that for all $\xi \in V$ we have $v_j(\xi) \in \{1, \ldots, 2\mu - 1\}$ $(j = 1, \ldots, n-k)$ satisfying that if*

$$z = (z', z'') \in D'_{\varepsilon,2\mu-2}(\xi) \times \mathbb{C}^{n-k} \text{ and } |z_{k+j} - \xi_{k+j}| = v_j(\xi) \tag{3.1}$$

for some $j = 1, \ldots, n-k$, then $z \notin V$.

Proof. It is sufficient to prove that $|Q_1(z)| \geq 1/2$ for z satisfying (3.1). Factorizing Q_1, we have

$$Q_1(\xi_1, \ldots, \xi_k, z_{k+1}) = (z_{k+1} - \alpha_1(\xi')) \cdots (z_{k+1} - \alpha_\mu(\xi')).$$

Then there exists $v_1(\xi) \in \{1, \ldots, 2\mu - 1\}$ such that for $|z_{k+1} - \xi_{k+1}| = v_1(\xi)$ we have $|z_{k+1} - \alpha_1(\xi')|, \ldots, |z_{k+1} - \alpha_\mu(\xi')| \geq 1$, and hence $|Q_1(\xi_1, \ldots, \xi_k, z_{k+1})| \geq 1$. In fact, we set $\{|\alpha_1(\xi') - \xi_{k+1}|, \ldots, |\alpha_\mu(\xi') - \xi_{k+1}|\} = \{\gamma_1, \ldots, \gamma_{\hat{\mu}}\}$ $(\hat{\mu} \leq \mu)$ as sets, and we assume that $\gamma_1 < \gamma_2 < \cdots < \gamma_{\hat{\mu}}$. Since $Q_1(\xi) = 0$, we have $\gamma_1 = 0$. Here we would like to find the minimal positive integer $v_1(\xi)$ satisfying $\gamma_\nu \leq v_1(\xi) - 1$ and $\gamma_{\nu+1} \geq v_1(\xi) + 1$ for some ν. For example, if $\gamma_2 \geq 2$, then we can take $v_1(\xi) = 1$. In the case where we have such ν, $v_1(\xi)$ is maximal if and only if $\gamma_2 \in (1, 2)$, $\gamma_3 \in (3, 4), \ldots$, $\gamma_{\hat{\mu}-1} \in (2\hat{\mu} - 5, 2\hat{\mu} - 4)$ and $\gamma_{\hat{\mu}} \geq 2\hat{\mu} - 2$. In this case, we can take $v_1(\xi) = 2\hat{\mu} - 3$. If there exists no such ν, that is, $\gamma_2 \in (1, 2)$, $\gamma_3 \in (3, 4), \ldots, \gamma_{\hat{\mu}-1} \in (2\hat{\mu} - 5, 2\hat{\mu} - 4)$ and $\gamma_{\hat{\mu}} \in (2\hat{\mu} - 3, 2\hat{\mu} - 2)$, then we take $v_1(\xi) = 2\hat{\mu} - 1$. Hence we can take $v_1(\xi) \in \{1, \ldots, 2\mu - 1\}$ satisfying the above condition.

Here we would like to take $\varepsilon \in (0, 1)$ so that if $|z_1 - \xi_1|, \ldots, |z_k - \xi_k| < \varepsilon(1 + |\xi|)^{-2\mu+2}$ and $|z_{k+1} - \xi_{k+1}| = v_1(\xi)$, then $|Q_1(z_1, \ldots, z_k, z_{k+1}) - Q_1(\xi_1, \ldots, \xi_k, z_{k+1})| \leq 1/2$. Let M be the maximum of moduli of all coefficients in $q_{1,1}, \ldots, q_{1,\mu}$. We can write

$$
\begin{aligned}
(3.2)\qquad |Q_1(z_1,\dots,z_k,z_{k+1}) - {}& Q_1(\xi_1,\dots,\xi_k,z_{k+1})| \\
&\le M\sum_{|\beta|\le 1} |z_1^{\beta_1}\cdots z_k^{\beta_k} - \xi_1^{\beta_1}\cdots\xi_k^{\beta_k}||z_{k+1}|^{\mu-1} \\
&\quad + \cdots + M\sum_{|\beta|\le\mu} |z_1^{\beta_1}\cdots z_k^{\beta_k} - \xi_1^{\beta_1}\cdots\xi_k^{\beta_k}|,
\end{aligned}
$$

where $\beta = (\beta_1,\dots,\beta_k)$ is a multi-index and $|\beta| = \beta_1 + \cdots + \beta_k$. Here we have the following estimates:
(1) Since $|z_{k+1} - \xi_{k+1}| = v_1(\xi)$,

$$|z_{k+1}| \le |\xi_{k+1}| + v_1(\xi) \le |\xi| + 2\mu - 1 \le (2\mu-1)(1+|\xi|).$$

(2) Since $|z_j|, |\xi| < |\xi| + \varepsilon(1+|\xi|)^{-2\mu+2} \le 1 + |\xi|$, we obtain

$$
\begin{aligned}
(3.3)\qquad & |z_1^{\beta_1}\cdots z_k^{\beta_k} - \xi_1^{\beta_1}\cdots\xi_k^{\beta_k}| \\
&\le |z_1^{\beta_1}\cdots z_k^{\beta_k} - z_1^{\beta_1}\cdots z_{k-1}^{\beta_{k-1}} z_k^{\beta_k-1}\xi_k| \\
&\quad + \cdots + |z_1\xi_1^{\beta_1-1}\xi_2^{\beta_2}\cdots\xi_k^{\beta_k} - \xi_1^{\beta_1}\cdots\xi_k^{\beta_k}| \\
&= |z_k-\xi_k||z_1^{\beta_1}\cdots z_{k-1}^{\beta_{k-1}} z_k^{\beta_k-1}| + \cdots + |z_1-\xi_1||\xi_1^{\beta_1-1}\xi_2^{\beta_2}\cdots\xi_k^{\beta_k}| \\
&\le |\beta|\varepsilon(1+|\xi|)^{-2\mu+2}(1+|\xi|)^{|\beta|-1} \\
&\le \mu\varepsilon(1+|\xi|)^{-\mu+1},
\end{aligned}
$$

where the number of terms in (3.3) is $|\beta|$.
(3) The number of terms in $\sum_{|\beta|\le\nu} |z_1^{\beta_1}\cdots z_k^{\beta_k} - \xi_1^{\beta_1}\cdots\xi_k^{\beta_k}||z_{k+1}|^{\mu-\nu}$ is bounded from above by

$$1 + k + \cdots + k^j \le 1 + k + \cdots + k^\mu \le (\mu+1)k^\mu.$$

It follows from (3.2) and these estimates that

$$|Q_1(z_1,\dots,z_k,z_{k+1}) - Q_1(\xi_1,\dots,\xi_k,z_{k+1})| \le M\mu^2(\mu+1)^{\mu-1}k^\mu\varepsilon.$$

Hence, we set

$$\varepsilon = \frac{1}{2M\mu^2(\mu+1)^{\mu-1}k^\mu},$$

and then the lemma holds for all $\xi \in V$. □

For simplification, we fix $\xi \in V$ and put $D' = D'_{\varepsilon,2\mu-2}(\xi)$, $D'' = \{z'' \in \mathbb{C}^{n-k} : |z_{k+j} - \xi_{k+j}| < v_j(\xi)$ for all $j = 1,\dots,n-k\}$ and $D = D'\times D''$. By Lemma 3.2, $\pi|_{D\cap V}$:

$D \cap V \to D'$ is proper. It follows from the normalization theorem that $D' \setminus \Delta^{-1}(0)$ is connected and

$$\pi|_{(V\cap D)\setminus\Delta^{-1}(0)} : (V \cap D) \setminus \Delta^{-1}(0) \to D' \setminus \Delta^{-1}(0)$$

is a $\tilde{\mu}$-fold covering mapping with $1 \le \tilde{\mu} \le \mu$. For $z' \in D' \setminus \Delta^{-1}(0)$, by renumbering $\alpha_1(z'), \dots, \alpha_\mu(z')$ we have $\alpha_1(z'), \dots, \alpha_{\tilde{\mu}}(z') \in \{z_{k+1} \in \mathbb{C} : |z_{k+1} - \xi_{k+1}| < v_1(\xi)\}$. Since symmetric polynomials of $\alpha_1, \dots, \alpha_{\tilde{\mu}}$ are bounded holomorphic functions in $D' \setminus \Delta^{-1}(0)$, it follows from Riemann's Extension Theorem that they extend to holomorphic functions in D'. Hence

$$\Delta'(z') = \prod_{1\le j<j'\le\tilde{\mu}} (\alpha_j(z') - \alpha_{j'}(z'))^2$$

is holomorphic in D'.

Let $\pi^{-1}(z') \cap V \cap D = \{\tau_1(z'), \dots, \tau_{\tilde{\mu}}(z')\}$ as sets such that

$$\{(\tau_1(z'))_{k+1}, \dots, (\tau_{\tilde{\mu}}(z'))_{k+1}\} = \{\alpha_1(z'), \dots, \alpha_{\tilde{\mu}}(z')\}$$

for $z' \in D' \setminus \Delta^{-1}(0)$, where $(\tau_j(z'))_{k+1}$ $(1 \le j \le \tilde{\mu})$ denote the $(k+1)$-th coordinate of $\tau_j(z')$. Then there exist $\varphi_0(z'), \dots, \varphi_{\tilde{\mu}-1}(z') \in \mathbb{C}$ uniquely such that

$$f(\tau_j(z')) = \varphi_0(z') + \varphi_1(z')\alpha_j(z') + \cdots + \varphi_{\tilde{\mu}-1}(z')\alpha_j(z')^{\tilde{\mu}-1} \tag{3.4}$$

for all $j = 1, \dots, \tilde{\mu}$ and $z' \in D' \setminus \Delta^{-1}(0)$. In fact, if we think (3.4) to be a system of linear equations in $\varphi_0(z'), \dots, \varphi_{\tilde{\mu}-1}(z')$, the determinant $W(z')$ of its coefficient matrix $\mathcal{A}$ is given by

$$\begin{aligned} W(z') &= \det \begin{pmatrix} 1 & \cdots & 1 \\ \alpha_1(z') & \cdots & \alpha_{\tilde{\mu}}(z') \\ \vdots & & \vdots \\ \alpha_1(z')^{\tilde{\mu}-1} & \cdots & \alpha_{\tilde{\mu}}(z')^{\tilde{\mu}-1} \end{pmatrix} \\ &= \prod_{1\le j<j'\le\tilde{\mu}} (\alpha_j(z') - \alpha_{j'}(z')) \ne 0, \quad \forall z' \in D' \setminus \Delta^{-1}(0). \end{aligned}$$

Then $W(z')^2 = \Delta'(z')$ and Cramer's rule gives

$$W(z')\varphi_j(z') = \sum_{l=1}^{\tilde{\mu}} T_{l,j}(z') f(\tau_l(z'))$$

for all $j = 0, \dots, \tilde{\mu} - 1$, where $(T_{l,j})_{l=1,\dots,\tilde{\mu};j=0,\dots,\tilde{\mu}-1}$ is the cofactor matrix of $\mathcal{A}$. It follows from the normalization theorem (2) that

$$|\alpha_l(z')| \le C_0(1 + |z'|) \tag{3.5}$$

for all $z' \in D'$ and $l = 1, \dots, \tilde{\mu}$. Thus, we have

Lemma 3.4. *There exist $C_3 > 0$ and $\omega \in \mathbb{N}$ depending only on Q_1 such that*

$$|\Delta'(z')\varphi_j(z')| \leq C_3 M_D(f)(1 + |z'|)^{\omega}$$

for all $z' \in D' \setminus \Delta^{-1}(0)$ and $j = 0, \dots, \tilde{\mu} - 1$, where $M_D(f) = \sup\{|f(z)| : z \in V \cap D\}$.

For the other roots $\alpha_{\tilde{\mu}+1}(z'), \dots, \alpha_{\mu}(z')$ of Q_1, setting

$$\Delta''(z') = \prod_{\substack{1 \leq j \leq \mu \\ \tilde{\mu}+1 \leq j' \leq \mu}} (\alpha_j(z') - \alpha_{j'}(z'))^2,$$

we have $\Delta = \Delta'\Delta''$. Since (3.5) hold for $l = \tilde{\mu} + 1, \dots, \mu$, we obtain $C_4 > 0$ satisfying

$$\begin{aligned} |\Delta''(z')| &\leq C_4(1 + |z'|)^{\mu(\mu-1) - \tilde{\mu}(\tilde{\mu}-1)} \\ &\leq C_4(1 + |z'|)^{\mu(\mu-1)}. \end{aligned}$$

Hence there exist $C_5 > 0$ and $\omega' \in \mathbb{N}$ independent of $\tilde{\mu}$ such that

$$|\Delta(z')\varphi_j(z')| \leq C_5 M_D(f)(1 + |z'|)^{\omega'}$$

for all $z' \in D' \setminus \Delta^{-1}(0)$. In particular, all $\Delta\varphi_j$ are bounded holomorphic functions. By Riemann's extension theorem, they extend to holomorphic functions in D'.

Since p is a weight function, we have A', $B' > 0$ indepedent of ξ satisfying

$$M_D(f) \leq A' \exp(B' p(\xi)).$$

Set

$$F(z) = \Delta(z')\varphi_0(z') + \Delta(z')\varphi_1(z')z_{k+1} + \cdots + \Delta(z')\varphi_{\tilde{\mu}-1}(z')z_{k+1}^{\tilde{\mu}-1}.$$

By the definition of weight functions, there exist A_1, $B_1 > 0$ independent of ξ such that

$$\begin{aligned} |F(z)| &\leq |\Delta(z')\varphi_0(z')| + |\Delta(z')\varphi_1(z')||z_{k+1}| + \cdots + |\Delta(z')\varphi_{\tilde{\mu}-1}(z')||z_{k+1}|^{\tilde{\mu}-1} \\ &\leq \tilde{\mu} C_5 A' \exp(B' p(\xi))(1 + |z|)^{\omega' + \tilde{\mu} - 1} \\ &\leq A_1 \exp(B_1 p(z)) \end{aligned}$$

for all $z \in D_{\varepsilon, 2\mu-2}(\xi)$. Finally, it follows from (3.4) that

$$F = \Delta f \tag{3.6}$$

in $(V \setminus \Delta^{-1}(0)) \cap D_{\varepsilon,2\mu-2}(\xi)$. Since $V \setminus \Delta^{-1}(0)$ is dense in V, (3.6) holds on $V \cap D_{\varepsilon,2\mu-2}(\xi)$. The proof of Lemma 3.2 is completed. $\square$

We next solve the Cousin first problem with estimates. We shall use some results from [9].

Lemma 3.5 ([9, Lemma 7.6.1]). *Let $d : \mathbb{R}^{2n} \to (0, 1]$ be a function such that*

$$d(x + y) \leq 2d(x), \quad \text{if } |y|_\infty = \max_{j=1,\dots,2n} |y_j| \leq 1. \tag{3.7}$$

Then there exist an open covering $\mathcal{U}^d = \{U_j^d\}_{j\in I(d)}$ of $\mathbb{R}^{2n}$ with open cubes U_j^d, a partition of unity $\chi_j^d \in C_0^\infty(U_j^d)$ and $C_6 > 0$ such that
(1) *$|x - y|_\infty \leq d(x)$ for all $x, y \in U_j^d$ and $j \in I(d)$;*
(2) *$\sharp\{j' \in I(d) : U_{j'}^d \cap U_j^d \neq \emptyset\} \leq 2^{8n}$ for all $j \in I(d)$.*
(3) *$|(\partial\chi_j/\partial x_\nu)(x)| \leq C_6/(d(x))$ for all $j \in I(d)$, $\nu = 1, \dots, 2n$ and $x \in \mathbb{R}^{2n}$.*
(4) *Let d' be another function satisfying* (3.7) *and $0 < d' \leq d$. There exists a refinement $\mathcal{U}^{d'}$ of $\mathcal{U}^d$ defined by a mapping $\rho_{d,d'} : I(d') \to I(d)$ with $\rho_{d,d''} = \rho_{d,d'} \circ \rho_{d',d''}$ satisfying* (1), (2) *and* (3). *Moreover, if $d' \leq \tilde{\varepsilon} d$, $\tilde{\varepsilon} < 1/64$, $j' \in I(d')$, $j = \rho_{d,d'}(j')$ and $x \in U_{j'}^{d'}$, then*

$$U_{j'}^{d'} \subset \{y \in \mathbb{R}^{2n} : |y - x|_\infty < \tilde{\varepsilon} d(x)\}$$

and

$$U_j^d \supset \left\{ y \in \mathbb{R}^{2n} : |y - x|_\infty < \left(\frac{1}{64} - \tilde{\varepsilon}\right) d(x) \right\}.$$

For $J = (j_0, \dots, j_\sigma) \in I(d)^{\sigma+1}$ we denote $U_J^d = U_{j_0}^d \cap \cdots \cap U_{j_\sigma}^d$. Let c be a cochain in $C^\sigma(\mathcal{U}^d, \mathcal{O})$ and let φ be a plurisubharmonic function in $\mathbb{C}^n$. Then we write

$$\|c\|_\varphi^2 = \sum_{J\in I(d)^{\sigma+1}} \int_{U_J^d} |c_J|^2 \exp(-\varphi) d\lambda.$$

We also define a coboundary operator $\delta : C^\sigma(\mathcal{U}^d, \mathcal{O}) \to C^{\sigma+1}(\mathcal{U}^d, \mathcal{O})$ by

$$(\delta c)_{J\in I(d)^{\sigma+2}} = \sum_{\nu=0}^{\sigma+1} (-1)^\nu c_{(j_0,\dots,\check{j}_\nu,\dots,j_{\sigma+1})}.$$

Lemma 3.6 ([9, Proposition 7.6.2]). *Let $-\log d$ be a plurisubharmonic function on $\mathbb{C}^n$. For every $c \in C^\sigma(\mathcal{U}^d, \mathcal{O})$ $(\sigma > 0)$ with $\delta c = 0$ and $\|c\|_\varphi < \infty$, we can find a cochain $c' \in C^{\sigma-1}(\mathcal{U}^d, \mathcal{O})$ such that $\delta c' = c$ and $\|c'\|_\psi \leq K_1 \|c\|_\varphi$, where ψ is a plurisubharmonic function in $\mathbb{C}^n$ defined by*

$$\psi(z) = \varphi(z) - \sigma \log d(z) + 2\log(1 + |z|^2),$$

and K_1 *is a constant independent of* φ, d *and* c.

Let

$$P = \begin{pmatrix} P_{1,1} & \cdots & P_{1,T} \\ \vdots & & \vdots \\ P_{\Lambda,1} & \cdots & P_{\Lambda,T} \end{pmatrix}$$

be a matrix with polynomial elements. Then P defines the sheaf homomorphism

$$P : \mathcal{O}^T \ni g \mapsto Pg \in \mathcal{O}^\Lambda. \tag{3.8}$$

Lemma 3.7 ([9, Lemma 7.6.3]). *The kernel* $\ker P$ *of the sheaf homomorphism* (3.8) *is generated by the germs of a finite number of* $Q_s = (Q_{1,s}, \ldots, Q_{T,s}) \in \mathbb{C}[z_1, \ldots, z_n]^T$ $(s = 1, \ldots, S)$ *satisfying*

$$\sum_{t=1}^{T} P_{\lambda,t} Q_{t,s} = 0$$

for all $\lambda = 1, \ldots, \Lambda$ *and* $s = 1, \ldots, S$.

Lemma 3.8 ([9, Lemma 7.6.4]). *Let* Ω *be a pseudoconvex domain and let* P *and* Q *be matrixes in* Lemma 3.7. *Then if* $g = (g_1, \ldots, g_T) \in \mathcal{O}(\Omega)^T$ *satisfies*

$$\sum_{t=1}^{T} P_{\lambda,t} g_t = 0$$

for all $\lambda = 1, \ldots, \Lambda$, *there exists* $h = (h_1, \ldots, h_S) \in \mathcal{O}(\Omega)^S$ *such that*

$$g_t = \sum_{s=1}^{S} Q_{t,s} h_s$$

for all $t = 1, \ldots, T$. *In particular,* $\ker P = \operatorname{Im} Q$ *holds.*

By putting $\Lambda = 1$, Lemmas 3.7 and 3.8 imply that $\mathcal{O}(\Omega)$ is a flat $\mathbb{C}[z_1, \ldots, z_n]$-module. This fact will play an important role later.

The following lemma gives estimates of solutions of the equation $Pv = u$ for $u \in \operatorname{Im} P$:

Lemma 3.9 ([9, Lemma 7.6.5]). *Let* Ω *be a neighborhood of* $0 \in \mathbb{C}^n$. *Then we have a neighborhood* Ω' *of* $0 \in \mathbb{C}^n$ *and constants* C_7, N_1 *satisfying that for all* $\eta \in (0, 1)$, $z \in \mathbb{C}^n$ *and* $u \in \mathcal{O}(\eta\Omega + \{z\})^T$, *there exists* $v \in \mathcal{O}(\eta\Omega' + \{z\})^T$ *such that* $Pv = Pu$

and

$$\sup_{\eta\Omega'+\{z\}} |v| \le C_7(1+|z|)^{N_1}\eta^{-N_1} \sup_{\eta\Omega+\{z\}} |Pu|.$$

Here $\eta\Omega+\{z\}=\{\eta w+z : w\in\Omega\}$.

We now prove a lemma important to solve the Cousin first problem with estimates. Let $P:\mathcal{O}^T\to\mathcal{O}^\Lambda$ be the sheaf homomorphism as above. Then $\mathcal{M}_P=\operatorname{Im}P$ is a subsheaf of $\mathcal{O}^\Lambda$ generated by $(P_{1,t},\ldots,P_{\Lambda,t})$ for $t=1,\ldots,T$. We denote by $C^\sigma(\mathcal{U}^d,\mathcal{M}_P,p)$ the set of cochains $c=\{c_J\}_{J\in I(d)^{\sigma+1}}\in C^\sigma(\mathcal{U}^d,\mathcal{M}_P)$ satisfying

$$\|c\|_p^2=\sum_{J\in I(d)^{\sigma+1}}\int_{U_J^d}|c_J|^2\exp(-p)d\lambda<\infty.$$

Lemma 3.10 (cf. [9, Lemma 7.6.10]). *We assume that* $-\log d$ *is a plurisubharmonic function. Then we have* N_2, $K_2>0$ *and* $\varepsilon_0<1/192$ *satisfying that for all* $c\in C^\sigma(\mathcal{U}^d,\mathcal{M}_P,p)$ $(\sigma>0)$ *with* $\delta c=0$, *there exists* $c'\in C^{\sigma-1}(\mathcal{U}^{\varepsilon_0 d},\mathcal{M}_P,p_{N_2})$ *such that* $\delta c'=\rho_{d,\varepsilon_0 d}{}^*c$ *and*

$$\|c'\|_{p_{N_2}}\le K_2\|c\|_p,$$

where $p_{N_2}(z)=N_2(p(z)-\log d(z)+\log(1+|z|^2))$.

Proof. Applying Lemma 3.9 for $\Omega:=\{z\in\mathbb{C}^n : |z|_\infty<1\}$, we have $r\in(0,1)$ and constants C_7, N_1 satisfying for all $\eta\in(0,1)$, $\xi\in\mathbb{C}\ n$ and $u\in\mathcal{O}(\eta\Omega+\{\xi\})^T$, there exists $v\in\mathcal{O}(\eta\Omega'+\{\xi\})^T$ such that $Pv=Pu$ and

$$\sup_{\eta\Omega'+\{\xi\}}|v|\le C_7(1+|\xi|)^{N_1}\eta^{-N_1}\sup_{\eta\Omega+\{\xi\}}|Pu|, \tag{3.9}$$

where $\Omega'=\{z\in\mathbb{C}^n : |z|_\infty<r\}$. For $\tilde\varepsilon<1/128$, it follows from Lemma 3.5 (4) that if $j'\in I(\tilde\varepsilon d)$, $j=\rho_{d,\tilde\varepsilon d}(j')$ and $\xi\in U_{j'}^{\tilde\varepsilon d}$, then

$$U_{j'}^{\tilde\varepsilon d}\subset\tilde\varepsilon d(\xi)\Omega+\{\xi\}\subset\left(\frac{1}{64}-\tilde\varepsilon\right)\Omega+\{\xi\}\subset U_j^d. \tag{3.10}$$

Here defining $\tilde\varepsilon:=r/(128(2+r))$ $(\le 1/384)$ and $\eta:=(1/128-\tilde\varepsilon/2)d(\xi)$, we have $\tilde\varepsilon d(\xi)<r\eta$, hence (3.10) implies that

$$U_{j'}^{\tilde\varepsilon d}\subset r\eta\Omega+\{\xi\}=\eta\Omega'+\{\xi\}. \tag{3.11}$$

On the other hand, we have $\eta<(1/96)d(\xi)<((1/64)-\tilde\varepsilon)d(\xi)$, that is,

$$\eta\Omega+\{\xi\}\subset\subset\frac{1}{96}d(\xi)\Omega\subset\left(\frac{1}{64}-\tilde\varepsilon\right)d(\xi)\Omega\subset U_j^d. \tag{3.12}$$

Then for $J' = (j'_0, \dots, j'_\sigma) \in I(\tilde{\varepsilon}d)^{\sigma+1}$ $J = \rho_{d,\tilde{\varepsilon}d}(J') := (\rho_{d,\tilde{\varepsilon}d}(j'_0), \dots, \rho_{d,\tilde{\varepsilon}d}(j'_\sigma))$ and $\xi \in U^{\tilde{\varepsilon}d}_{J'}$, we obtain from (3.11) and (3.12)

$$U^{\tilde{\varepsilon}d}_{J'} \subset \eta\Omega' + \{\xi\} \subset \eta\Omega + \{\xi\} \subset\subset \frac{1}{96}d(\xi)\Omega + \{\xi\} \subset U^d_J. \tag{3.13}$$

Hence it follows from (3.9) that for all $u \in \mathcal{O}(U^d_J)^T$ $(\subset \mathcal{O}(\eta\Omega + \{\xi\})^T)$ there exists $v \in \mathcal{O}(\eta\Omega' + \{\xi\})^T$ such that $Pv = Pu$ and

$$\sup_{U^{\tilde{\varepsilon}d}_{J'}} |v| \le C_7(1 + |\xi|)^{N_1}\eta^{-N_1} \sup_{\eta\Omega+\{\xi\}} |Pu|. \tag{3.14}$$

By [9, Theorem 2.2.3], (3.12) implies that there exists $C_8 > 0$ independent of ξ such that

$$\sup_{\eta\Omega+\{\xi\}} |g| \le C_8 \|g\|_{L^1((1/96)d(\xi)\Omega+\{\xi\})}$$

for all $g \in \mathcal{O}(U^d_J)$. It follows from Schwarz's inequality that

$$\begin{aligned}\sup_{\eta\Omega+\{\xi\}} |Pu| \;&\le\; C_8 \left(\|(Pu)_1\|_{L^1((1/96)d(\xi)\Omega+\{\xi\})} + \cdots + \|(Pu)_\Lambda\|_{L^1((1/96)d(\xi)\Omega+\{\xi\})}\right) \\ &\le\; \Lambda C_8 \left(\int_{(1/96)d(\xi)\Omega+\{\xi\}} |Pu|^2 d\lambda\right)^{1/2} \le \Lambda C_8 \left(\int_{U^d_J} |Pu|^2 d\lambda\right)^{1/2}.\end{aligned}$$

Since p is a weight, by Lemma 3.5 (1) there exist C'_1, $C'_2 > 0$ independent of d and J such that $p(z') \le C'_1 p(z) + C'_2$ for z, $z' \in U^d_J$. Then we obtain

$$\exp(-C'_1 p(\xi)) \int_{U^d_J} |Pu(z)|^2 d\lambda(z) \le e^{C'_2} \int_{U^d_J} |Pu(z)|^2 \exp(-p(z)) d\lambda(z).$$

Hence it follows from (3.7) that

$$|v(\xi)|^2(1 + |\xi|^2)^{-2N_1} d(\xi)^{2N_1} \exp(-C'_1 p(\xi)) \le C_9 \int_{U^d_J} |Pu(z)|^2 \exp(-p(z)) d\lambda(z),$$

where $C_9 = \Lambda C_7 C_8 2^{2N_1}(1/128 - \tilde{\varepsilon}/2)^{-2N_1} e^{C'_2}$. Therefore putting $N'_2 = \max\{N_1, C'_1\}$, we obtain

$$\int_{U^{\tilde{\varepsilon}d}_{J'}} |v(\xi)|^2 \exp(-p_{N'_2}(\xi)) d\lambda(\xi) \le C_9 \int_{U^d_J} |Pu(z)|^2 \exp(-p(z)) d\lambda(z). \tag{3.15}$$

We prove this lemma by induction for decreasing σ. Note that it is valid when $\sigma = 2^{8n} + 1$, since $C^\sigma(\mathcal{U}, \cdot) = \{0\}$ by Lemma 3.5 (2). We assume that it have been proved for all P when σ is replaced by $\sigma + 1$. By [9, Lemma 7.2.9], there exists $\gamma \in$

$C^\sigma(\mathcal{U}^d, \mathcal{O}^T)$ such that $c_J = P\gamma_J$ for all $J \in I(d)^{\sigma+1}$. To obtain contorol of γ_J we pass to the refinement $\mathcal{U}^{\tilde{\varepsilon}d}$ for which (3.15) is applicable. Then we can choose $\gamma'_{J'} \in \mathcal{O}(U^{\tilde{\varepsilon}d}_{J'})^T$ $(J' \in I(\tilde{\varepsilon}d)^{\sigma+1})$ so that with $J = \rho_{d,\tilde{\varepsilon}d}J'$ we have

$$P\gamma'_{J'} = P\gamma_J = c_J \tag{3.16}$$

in $U^{\tilde{\varepsilon}d}_{J'}$ and

$$\int_{U^{\tilde{\varepsilon}d}_{J'}} |\gamma'_{J'}|^2 \exp(-p_{N'_2})d\lambda \le C_9 \int_{U^d_J} |c_J|^2 \exp(-p)d\lambda.$$

Here we need to calculate $\sharp\rho^{-1}_{d,\tilde{\varepsilon}d}(J)$ to give the estimate of $\|\gamma'\|_{p_{N'_2}}$. For the refinement $\mathcal{U}^{\tilde{\varepsilon}^2 d}$ of $\mathcal{U}^{\tilde{\varepsilon}d}$, it follows from Lemma 3.5 (4) that

$$U^{\tilde{\varepsilon}d}_{J'} \supset \left\{ z \in \mathbb{C}^n : |z - \xi|_\infty < \tilde{\varepsilon}\left(\frac{1}{64} - \tilde{\varepsilon}\right) d(\xi) \right\}$$

for $\xi \in U^{\tilde{\varepsilon}^2 d}_{J''}$ and $J' = \rho_{\tilde{\varepsilon}d,\tilde{\varepsilon}^2 d}(J'')$. On the other hand, we know

$$U^d_J \subset \{z \in \mathbb{C}^n : |z - \xi|_\infty < d(\xi)\}.$$

Hence it follows from Lemma 3.5 (2) that

$$\sharp\rho^{-1}_{d,\tilde{\varepsilon}d}(J) \le 2^{8n}\left(\frac{\tilde{\varepsilon}}{32} - 2\tilde{\varepsilon}^2\right)^{-2n} =: C_{10}.$$

Thus we obtain

$$\begin{aligned} \|\gamma'\|^2_{p_{N'_2}} &= \sum_{J' \in I(\tilde{\varepsilon}d)^{\sigma+1}} \int_{U^{\tilde{\varepsilon}d}_{J'}} |\gamma'_{J'}|^2 \exp(-p_{N'_2})d\lambda \\ &\le C_{10} \sum_{J \in I(d)^{\sigma+1}} C_9 \int_{U^d_J} |c_J|^2 \exp(-p)d\lambda \\ &= C_{10}C_9\|c\|^2_p. \end{aligned} \tag{3.17}$$

It also follows from (3.16) that $P\gamma' = \rho_{d,\tilde{\varepsilon}d}{}^* c$. Since $\delta c = 0$ and P is defined globally, we have $P\delta\gamma' = \delta P\gamma' = 0$. Thus $\delta\gamma' = \gamma''$ belongs to $C^{\sigma+1}(\mathcal{U}^{\tilde{\varepsilon}d}, \ker P, p_{N'_2})$, and $\delta\gamma'' = 0$. If we choose a $T \times S$ matrix Q as in Lemma 3.8, it follows that $\ker P = \operatorname{Im} Q = \mathcal{M}_Q$, so the inductive hypothesis can be applied. It shows that we can find $\hat{\epsilon} < \tilde{\epsilon}$, $N''_2 > N'_2$ and $K'_2 > 0$ such that for some $\gamma''' \in C^\sigma(\mathcal{U}^{\hat{\epsilon}d}, \ker P, p_{N''_2})$ we have $\|\gamma'''\|_{p_{N''_2}} \le K'_2\|\gamma''\|_{p_{N'_2}}$ and $\delta\gamma''' = \rho_{\tilde{\varepsilon}d,\hat{\varepsilon}d}{}^*\gamma''$.

Setting $\tilde{\gamma} = \rho_{\tilde{\varepsilon}d,\hat{\varepsilon}d}{}^*\gamma' - \gamma''' \in C^\sigma(\mathcal{U}^{\hat{\varepsilon}d}, \mathcal{O}^T)$, we have $\delta\tilde{\gamma} = \rho_{\tilde{\varepsilon}d,\hat{\varepsilon}d}{}^*\gamma'' - \delta\gamma''' = 0$, and for some C_{11} independent of c we have $\|\tilde{\gamma}\|_{p_{N''_2}} \le C_{11}\|c\|_p$ by the same method that we have proved (3.17). Hence Lemma 3.6 shows that for some $N_2''' > 0$ we can find $\hat{\gamma} \in C^{\sigma-1}(\mathcal{U}^{\hat{\varepsilon}d}, \mathcal{O}^T)$ so that $\tilde{\gamma} = \delta\hat{\gamma}$ and $\|\hat{\gamma}\|_{p_{N_2'''}} \leq K_1\|\tilde{\gamma}\|_{p_{N_2''}} \leq K_1 C_{11}\|c\|_p$. If we set $c' = P\hat{\gamma}$, it follows that

$$\delta c' = P\delta\hat{\gamma} = P\tilde{\gamma} = P\rho_{\tilde{\varepsilon}d,\hat{\varepsilon}d}{}^*\gamma' - P\gamma''' = \rho_{\tilde{\varepsilon}d,\hat{\varepsilon}d}{}^* P\gamma' = \rho_{\tilde{\varepsilon}d,\hat{\varepsilon}d}{}^*\rho_{d,\tilde{\varepsilon}d}{}^* c = \rho_{d,\hat{\varepsilon}d}{}^* c.$$

Finally, it is clear that there exists N_2, $K_2 > 0$ such that $\|c'\|_{p_{N_2}} \leq K_2\|c\|_p$, because it is sufficient to consider the estimate about P. The proof of the lemma is finished. □

We shall apply Lemma 3.10 to the following settings. Put

$$d_V(z) = \frac{\varepsilon}{2\sqrt{2}}(2\sqrt{2n}(2\mu-2) + |z|)^{2-2\mu},$$

where ε and μ are decided before.

Lemma 3.11. *d_V has the following properties*:
(1) *If $w \in \mathbb{C}^n$ and $|w|_\infty \leq 1$, then we have $d_V(z+w) \leq 2d_V(z)$ for all $z \in \mathbb{C}^n$. Hence there exists an open covering $\mathcal{U}^{d_V} = \{U_j^{d_V}\}_{j \in I(d_V)}$ satisfying* Lemma 3.5.
(2) *$-\log d_V(z)$ is a plurisubharmonic function.*
(3) *If $U_j^{d_V} \in \mathcal{U}^{d_V}$ and $U_j^{d_V} \cap V \neq \emptyset$, then $U_j^{d_V} \subset D_{\varepsilon,2\mu-2}(\xi)$ holds for every $\xi \in U_j^{d_V} \cap V$.*

Proof. The lemma is clear when $\mu = 1$, so we assume that $\mu \geq 2$.
(1) If $w \in \mathbb{C}^n$ and $|w|_\infty \leq 1$, then $|z| \leq |z+w| + |w| \leq |z+w| + \sqrt{2n}$. Hence we have

$$\begin{aligned}
d_V(z) &= \frac{\varepsilon}{2\sqrt{2}}(2\sqrt{2n}(2\mu-2))^{2-2\mu}\left(1 + \frac{|z|}{2\sqrt{2n}(2\mu-2)}\right)^{2-2\mu} \\
&\geq \frac{\varepsilon}{2\sqrt{2}}(2\sqrt{2n}(2\mu-2))^{2-2\mu}\left(1 + \frac{|z+w|}{2\sqrt{2n}(2\mu-2)} + \frac{1}{2(2\mu-2)}\right)^{2-2\mu} \\
&\geq \frac{\varepsilon}{2\sqrt{2}}(2\sqrt{2n}(2\mu-2))^{2-2\mu}\left(1 + \frac{|z+w|}{2\sqrt{2n}(2\mu-2)}\right)^{2-2\mu}\left(1 + \frac{1}{2(2\mu-2)}\right)^{2-2\mu} \\
&\geq \frac{1}{2} \cdot \frac{\varepsilon}{2\sqrt{2}}(2\sqrt{2n}(2\mu-2))^{2-2\mu}\left(1 + \frac{|z+w|}{2\sqrt{2n}(2\mu-2)}\right)^{2-2\mu} \\
&\geq \frac{1}{2}d_V(z+w),
\end{aligned}$$

since $(1 + 1/2\nu)^{-\nu} \searrow \exp(-1/2) > 1/2$ as $\nu \to \infty$.

(2) is clear.

(3) Fix $\xi \in U_j^{d_V} \cap V$. It follows from Lemma 3.5 (1) that $|z - \xi|_\infty \leq d_V(\xi)$ for all

$z \in U_j^{d_V}$. Hence we obtain

$$|z_j - \xi_j| \leq \frac{\varepsilon(2\sqrt{2n}(2\mu - 2) + |\xi|)^{2-2\mu}}{2} < \varepsilon(1 + |\xi|)^{2-2\mu}$$

for all $j = 1, \ldots, n$, so $z \in D_{\varepsilon,2\mu-2}(\xi)$. $\square$

Since the polynomial ring $\mathbb{C}[z_1, \ldots, z_n]$ is Noetherian, the prime ideal I_V is finitely generated by $P_1, \ldots, P_T \in \mathbb{C}[z_1, \ldots, z_n]$. Let $P = (P_1, \ldots, P_T)$ be a $1 \times T$ matrix.

Lemma 3.12. *There exists $\tilde{F} \in A_p(\mathbb{C}^n)$ such that $\tilde{F}|_V \equiv \Delta f$.*

Proof. It follows from Lemma 3.2 and Lemma 3.11 (3) that if $U_j^{d_V} \in \mathcal{U}^{d_V}$ and $U_j^{d_V} \cap V \neq \emptyset$, then there exist $\xi \in V$ and $F^j \in \mathcal{O}(D_{\varepsilon,2\mu-2}(\xi))$ such that $\Delta f - F^j = 0$ on $V \cap D_{\varepsilon,2\mu-2}(\xi)$ and

$$|F^j(z)| \leq A_1 \exp(B_1 p(z)) \tag{3.18}$$

for every $z \in D_{\varepsilon,2\mu-2}(\xi)$. We also put $F^j = 0$, when $U_j^{d_V} \cap V = \emptyset$. We would like to apply Lemma 3.10 for $\sigma = 1$. Defining $c \in C^1(\mathcal{U}^{d_V}, \mathcal{O})$ by $c_{(j_0,j_1)} = F^{j_0} - F^{j_1}$, we have $F^{j_0} - F^{j_1} = \Delta f - \Delta f = 0$ on $V \cap U_{j_0}^{d_V} \cap U_{j_1}^{d_V}$. It follows from (3.18) and Lemma 3.5 (2) that there exists $C_{12} > 0$ such that $\|c\|_{C_{12}p} < \infty$. On the other hand, it is clear that $\delta c = 0$, that is, $c \in C^1(\mathcal{U}^{d_V}, \mathcal{M}_P, C_{12}p)$. Hence Lemma 3.10 gives $\varepsilon_0 < 1/384$, N_2, $K_2 > 0$ and $c' \in C^0(\mathcal{U}^{\varepsilon_0 d_V}, \mathcal{M}_P, p_{N_2})$ so that $\delta c' = \rho_{d_V,\varepsilon_0 d_V}{}^* c$ and $\|c'\|_{p_{N_2}} \leq K_2\|c\|_{C_{12}p}$. It follows from the definition of weight functions that there exists $N_3 > 0$ such that $\|c'\|_{N_3 p} \leq K_2\|c\|_{C_{12}p}$. Here we put $\tilde{F} = F^j + c'_{j'}$ in $U_{j'}^{\varepsilon_0 d_V}$, where $j = \rho_{d_V,\varepsilon_0 d_V}(j')$. Then $\tilde{F}$ belongs to $\mathcal{O}(\mathbb{C}^n)$ and Lemma 2.2 (3) gives $\tilde{F} \in A_p(\mathbb{C}^n)$. $\square$

Here we make $\hat{F} \in A_p(\mathbb{C}^n)$ with the required properties from $\tilde{F} \in A_p(\mathbb{C}^n)$ made in Lemma 3.12. We shall use some result in the ring thoery. For an ideal $I \subset \mathbb{C}[z_1, \ldots, z_n]$, we set $\tilde{I} = \mathcal{O}(\mathbb{C}^n) \otimes_{\mathbb{C}[z_1,\ldots,z_n]} I = \mathcal{O}(\mathbb{C}^n)I$.

Lemma 3.13 (cf. [6, Lemma 3.5 in Chapter 8]). *For two ideals I_1 and I_2 in $\mathbb{C}[z_1, \ldots, z_n]$, $\widetilde{(I_1 \cap I_2)} = \tilde{I}_1 \cap \tilde{I}_2$.*

For $R \in \mathbb{C}[z_1, \ldots, z_n]$, set $(I : R) = \{g \in \mathbb{C}[z_1, \ldots, z_n] : Rg \in I\}$ and $(\tilde{I} : R) = \{\tilde{g} \in \mathcal{O}(\mathbb{C}^n) : R\tilde{g} \in \tilde{I}\}$.

Lemma 3.14 (cf. [6, Lemma 3.6 in Chapter 9]). *For an ideal $I \subset \mathbb{C}[z_1, \ldots, z_n]$, $(\tilde{I} : R) = \widetilde{(I : R)}$.*

Note that Lemmas 3.13 and 3.14 follow from the flatness of $\mathcal{O}(\mathbb{C}^n)$.

Lemma 3.15 (cf. [6, Lemma 3.13 in Chapter 8]). *Let $I \subset \mathbb{C}[z_1, \dots, z_n]$ be a primary ideal. Set $V_I = \{z \in \mathbb{C}^n : P(z) = 0$ for all $P \in I\}$. Then we have $(I : R) = I$, if $R|_{V_I} \not\equiv 0$.*

Proof. Since it is obvious that $I \subset (I : R)$, we have only to prove that $I \supset (I : R)$. For $P \in (I : R)$, it follows $RP \in I$. Assuming that $P \notin I$, we have $R^\nu \in I$ for some $\nu \in \mathbb{N}$, since I is a primary ideal. Hence it follows that $R|_{V_I} \equiv 0$, which is a contradiction. □

Here we can prove the following lemma by an argument similar to the proof of Lemma 3.12:

Lemma 3.16 (cf. [9, Theorem 7.6.11]). *Let $I \subset \mathbb{C}[z_1, \dots, z_n]$ be an ideal generated by $Q_1, \dots, Q_T$. If $g \in \tilde{I} \cap A_p(\mathbb{C}^n)$, then there exist $a_1, \dots, a_T \in A_p(\mathbb{C}^n)$ such that*

$$g = a_1 Q_1 + \cdots + a_T Q_T.$$

[9, Theorem 7.4.8] also implies that there exists $\tilde{f} \in \mathcal{O}(\mathbb{C}^n)$ with no growth conditions such that $\tilde{f}|_V \equiv f$.

Lemma 3.17. *We have $\hat{F} \in A_p(\mathbb{C}^n)$ satisfying that $\hat{F} - \tilde{f} \in \tilde{I}_V$, that is, $\hat{F}|_V \equiv f$.*

Proof. Let $J \in \mathbb{C}[z_1, \dots, z_n]$ be the ideal generated by $P_1, \dots, P_T$ and Δ. By Lemma 3.12, it follows that $\tilde{F} - \Delta \tilde{f} \in \tilde{I}_V$, that is, $\tilde{F} \in \tilde{J}$. Applying Lemma 3.16 to J, we have $a_1, \dots, a_T, b \in A_p(\mathbb{C}^n)$ satisfying that

$$\tilde{F} = a_1 P_1 + \cdots + a_T P_T + b\Delta.$$

Here if we set $\hat{F} = b$, then $\Delta\hat{F} - \Delta\tilde{f} = \Delta(\hat{F} - \tilde{f}) \in \tilde{I}_V$. Hence it follows from Lemmas 3.14 and 3.15 that

$$\hat{F} - \tilde{f} \in (\tilde{I}_V : \Delta) = \widetilde{(I_V : \Delta)} = \tilde{I}_V,$$

so that $\hat{F}|_V \equiv f$. □

Proof of Theorem 3.1. Let $V \subset \mathbb{C}^n$ be an algebraic subset. Then there exist a finite number of irreducible algebraic varieties $V_1, \dots, V_S$ such that $V = V_1 \cup \cdots \cup V_S$. We shall prove Theorem 3.1 by induction on S. When $S = 1$, we have already proved in Lemma 3.17. Here we can assume that $S = 2$, since the proofs for $S \geq 3$ are the same as for $S = 2$. Then we have $V = V_1 \cup V_2$ and $I_V = I_{V_1} \cap I_{V_2}$. For

$f \in A_p(V)$, it follows from [9, Theorem 7.4.8] that there exists $\tilde{f} \in \mathcal{O}(\mathbb{C}^n)$ with no growth conditions such that $\tilde{f}|_V \equiv f$. Since the theorem is valid for V_1 (resp. V_2), we have $\hat{F}_1 \in A_p(\mathbb{C}^n)$ (resp. $\hat{F}_2 \in A_p(\mathbb{C}^n)$) such that $\hat{F}_1|_{V_1} \equiv f$ (resp. $\hat{F}_2|_{V_2} \equiv f$). Let $P_1, \dots, P_{T_1}$ (resp. $Q_1, \dots, Q_{T_2}$) generate I_{V_1} (resp. I_{V_2}). If $J \subset \mathbb{C}[z_1, \dots, z_n]$ is the ideal generated by $P_1, \dots, P_{T_1}, Q_1, \dots, Q_{T_2}$, we have $I_{V_1} \cap I_{V_2} \subset J$. Since $\hat{F}_1 - \tilde{f} \in \tilde{I}_{V_1}$ and $\hat{F}_2 - \tilde{f} \in \tilde{I}_{V_2}$, it follows that

$$\hat{F}_1 - \hat{F}_2 = (\hat{F}_1 - \tilde{f}) - (\hat{F}_2 - \tilde{f}) \in \tilde{I}_{V_1} - \tilde{I}_{V_2} \subset \tilde{J}.$$

Applying Lemma 3.16 to J, we have $a_1, \dots, a_{T_1}, b_1, \dots, b_{T_2} \in A_p(\mathbb{C}^n)$ satisfying

$$\hat{F}_1 - \hat{F}_2 = a_1 P_1 + \cdots + a_{T_1} P_{T_1} + b_1 Q_1 + \cdots + b_{T_2} Q_{T_2}.$$

Here we set

$$\hat{F} = \hat{F}_1 - (a_1 P_1 + \cdots + a_{T_1} P_{T_1}) = \hat{F}_2 + (b_1 Q_1 + \cdots + b_{T_2} Q_{T_2}).$$

Then since $\hat{F}_1 - \tilde{f} \in \tilde{I}_{V_1}$ and $\hat{F}_2 - \tilde{f} \in \tilde{I}_{V_2}$, it follows from Lemma 3.13 that

$$\hat{F} - \tilde{f} \in \tilde{I}_{V_1} \cap \tilde{I}_{V_2} = (\widetilde{I_{V_1} \cap I_{V_2}}) = \tilde{I}_V,$$

so that $\hat{F}|_V \equiv f$. Thus the proof of Theorem 3.1 is finished. □

4. Proof of the main theorem

Applying Theorem A for $X = \{\zeta_\nu\}$, we have $f_1, \dots, f_m \in \mathcal{O}(\mathbb{C}^m)$ and constants $\varepsilon_1, C_3, A, B > 0$ with

$$|f_j(\zeta)| \leq A \exp(B|\zeta|^a) \tag{4.1}$$

for all $\zeta \in \mathbb{C}^m$ and $j = 1, \dots, m$,

$$Z(f_1, \dots, f_m) \supset X \tag{4.2}$$

and

$$\sum_{j=1}^{m} |D_u f_j(\zeta_\nu)| \geq \varepsilon_1 \exp(-C_3 |\zeta_\nu|^a) \tag{4.3}$$

for all $\nu \in \mathbb{N}$ and $u \in S^{2m-1}$. Fix $\nu \in \mathbb{N}$ and $u \in S^{2m-1}$. Set $\tilde{f}_{j,\nu,u}(w) = f_j(\zeta_\nu + wu)$, which is an entire function on $\mathbb{C}$. It follows from the chain rule that

$$\tilde{f}'_{j,\nu,u}(0) = \sum_{l=1}^{m} \frac{\partial f_j}{\partial \xi_l}(\zeta_\nu) \cdot u_l = D_u f_j(\zeta_\nu).$$

Hence from (4.3), there exists $j(\nu, u) \in \{1, \ldots, m\}$ such that

$$|\tilde{f}'_{j(\nu,u),\nu,u}(0)| \geq \frac{\varepsilon_1}{m} \exp(-C_3|\zeta_\nu|^a). \tag{4.4}$$

In the rest of the proof, we denote $\tilde{f}_{j(\nu,u),\nu,u}$ by $\tilde{f}_{j(\nu,u)}$. Put $Z_{\nu,u} = \{w \in \mathbb{C} : \tilde{f}_{j(\nu,u)}(w) = 0\}$, which contains 0 by (4.2), and $d_{\nu,u} = \min\{1, \mathrm{dist}(0, Z_{\nu,u} \setminus \{0\})\}$. From (4.1), we have $|f_{j(\nu,u),\nu,u}(\zeta_\nu + wu)| \leq A \exp(B|\zeta_\nu + wu|^a)$ for $|w| \leq 1$. Since $|(\zeta_\nu + wu) - \zeta_\nu| = |wu| = |w| \leq 1$ and $|\cdot|^a$ is a weight function, there exists A_1, $B_1 > 0$ independent of ν and u such that

$$|\tilde{f}_{j(\nu,u)}(w)| \leq A_1 \exp(B_1|\zeta_\nu|^a), \tag{4.5}$$

Set $g_{\nu,u}(w) = \tilde{f}_{j(\nu,u)}(w)/w$. Since $\tilde{f}_{j(\nu,u)}$ has zero of order only one at $w = 0$ by (4.2) and (4.4), we obtain $g_{\nu,u} \in A(\mathbb{C})$ and

$$g_{\nu,u}(0) = \tilde{f}'_{j(\nu,u)}(0) \neq 0. \tag{4.6}$$

It is satisfied for $|w| = 1$ that

$$|g_{\nu,u}(w)| = \frac{|\tilde{f}_{j(\nu,u)}(w)|}{|w|} = |\tilde{f}_{j(\nu,u)}(w)| \leq A_1 \exp(B_1|\zeta_\nu|^a).$$

Hence it follows from the Maximal Modulus Theorem that

$$|g_{\nu,u}(w)| \leq A_1 \exp(B_1|\zeta_\nu|^a) \tag{4.7}$$

for $|w| \leq 1$. We denote $G_{\nu,u} \in A(\mathbb{C})$ by

$$G_{\nu,u}(w) = \frac{g_{\nu,u}(w) - g_{\nu,u}(0)}{3A_1 \exp(B_1|\zeta_\nu|^a)}.$$

Then we have $G_{\nu,u}(0) = 0$ and (4.7) gives that $|G_{\nu,u}(w)| < 1$ for $|w| \leq 1$. Hence the Schwarz lemma implies that $|G_{\nu,u}(w)| \leq |w|$ for $|w| \leq 1$. In particular, for $\tilde{w} \in (Z_{\nu,u} \setminus \{0\}) \cap \{w \in \mathbb{C} : |w| < 1\}$, which is a zero of $g_{\nu,u}$ in $\{w \in \mathbb{C} : |w| < 1\}$, we have from (4.4) and (4.6)

$$|\tilde{w}| \geq |G_{\nu,u}(\tilde{w})| = \frac{|g_{\nu,u}(0)|}{3A_1 \exp(B_1|\zeta_\nu|^a)} = \frac{|\tilde{f}'_{j(\nu,u)}(0)|}{3A_1 \exp(B_1|\zeta_\nu|^a)} \geq \varepsilon_2 \exp(-C_4|\zeta_\nu|^a),$$

where ε_2 and C_4 are independent of ν and u. Thus the definition of $d_{\nu,u}$ gives that

$$d_{\nu,u} \geq \varepsilon_2 \exp(-C_4|\zeta_\nu|^a). \tag{4.8}$$

Now, we need the Borel-Carathèodory inequality. (For the proof, see e.g. [1, Corollary 4.5.10].)

Borel-Carathèodory inequality. *Let h be a function which is holomorphic in a neighborhood of $|w| \leq R$ and has no zero in $|w| < R$. If $h(0) = 1$ and $0 \leq |z| \leq r < R$, then the following estimate holds*:

$$\log |h(z)| \geq -\frac{2r}{R-r} \log \max_{|\omega|=R} |h(\omega)|.$$

Since $g_{\nu,u}(0) \neq 0$ from (4.6), we apply this inequality to $h(w) = g_{\nu,u}(w)/g_{\nu,u}(0)$, $R = d_{\nu,u}$ and $r = d_{\nu,u}/2$ to obtain

$$\begin{aligned}\log \left| \frac{g_{\nu,u}(w)}{g_{\nu,u}(0)} \right| &\geq -\frac{2 \cdot d_{\nu,u}/2}{d_{\nu,u} - d_{\nu,u}/2} \log \max_{|\omega|=d_{\nu,u}} \left| \frac{g_{\nu,u}(\omega)}{g_{\nu,u}(0)} \right| \\ &= -2 \log \max_{|\omega|=d_{\nu,u}} \left| \frac{g_{\nu,u}(\omega)}{g_{\nu,u}(0)} \right|\end{aligned}$$

for $|w| \leq d_{\nu,u}/2$. Then it follows from (4.4), (4.6) and (4.7) that

$$\begin{aligned}|g_{\nu,u}(w)| &\geq |g_{\nu,u}(0)| \left(\max_{|\omega|=d_{\nu,u}} \left| \frac{g_{\nu,u}(\omega)}{g_{\nu,u}(0)} \right| \right)^{-2} \\ &= |g_{\nu,u}(0)|^3 \left(\max_{|\omega|=d_{\nu,u}} |g_{\nu,u}(\omega)| \right)^{-2} \\ &\geq \varepsilon_3 \exp(-C_5 |\zeta_\nu|^a),\end{aligned} \tag{4.9}$$

where ε_3 and C_5 is independent of ν and u. Put $\hat{d}_\nu = \varepsilon_2 \exp(-C_4 |\zeta_\nu|^a)$, where ε_2 and C_4 are given in (4.8). Since $\hat{d}_\nu \leq d_{\nu,u}$ by (4.8), it follows from (4.9) that for $|w| = \hat{d}_\nu/2$ $|\tilde{f}_{j(\nu,u)}(w)| = |w \cdot g_{\nu,u}(w)| \geq \varepsilon_4 \exp(-C_6 |\zeta_\nu|^a)$, where ε_4 and C_6 is independent of ν and u. Thus we have proved that for every $u \in S^{2m-1}$, there exists $j(\nu,u) \in \{1, \ldots, m\}$ such that $|f_{j(\nu,u)}(\zeta_\nu + wu)| \geq \varepsilon_4 \exp(-C_6 |\zeta_\nu|^a)$ for $|w| = \hat{d}_\nu/2$. Hence we have

$$\begin{aligned}|f(\zeta_\nu + wu)| &= \left(\sum_{j=1}^m |f_j(\zeta_\nu + wu)|^2 \right)^{1/2} \geq |f_{j(\nu,u)}(\zeta_\nu + wu)| \\ &\geq \varepsilon_4 \exp(-C_6 |\zeta_\nu|^a)\end{aligned} \tag{4.10}$$

for $|w| = \hat{d}_\nu/2$.

We now consider $f \circ F : \mathbb{C}^n \to \mathbb{C}^m$. Since $\max_{j=1,\ldots,m} \deg F_j = d$ and $b \geq ad$, there exist α, $\beta > 0$ such that

$$|F(z)|^a \leq \alpha |z|^b + \beta \tag{4.11}$$

for all $z \in \mathbb{C}^n$. Then we have from (4.1) and (4.11)

$$|f_j \circ F(z)| \leq A \exp(B|F(z)|^a) \leq Ae^{\beta B} \exp(\alpha B |z|^b)$$

for all $z \in \mathbb{C}^n$ and $j = 1, \dots, m$, that is, $f \circ F \in A_{|\,.\,|^b}(\mathbb{C}^n)^m$.

Set $U_\nu = \{\xi \in \mathbb{C}^m : |\xi - \zeta_\nu| \le \hat{d}_\nu/2\}$. Denote by V_ν the connected component of $S_{|\,.\,|^a}(f; \varepsilon_4, C_6)$ including ζ_ν. Then (4.10) implies that $V_\nu \subset U_\nu$. We also have $U_\nu \cap (Z(f_1, \dots, f_m) \setminus \{\zeta_\nu\}) = \emptyset$. Namely, for $\xi \in Z(f_1, \dots, f_m) \setminus \{\zeta_\nu\}$ there exists $u \in S^{2m-1}$ such that $\xi = \zeta_\nu + |\xi - \zeta_\nu| u$. It follows from the definition of $d_{\nu,u}$ and (4.8) that $|\xi - \zeta_\nu| \ge d_{\nu,u} \ge \varepsilon_2 \exp(-C_4|\zeta_\nu|^a) = \hat{d}_\nu$, so that $\xi \notin U_\nu$. Now setting $\varepsilon_5 = \varepsilon_4 \exp(-\beta C_6)$ and $C_7 = \alpha C_6$, we claim that the union $\hat{V}_\nu$ of the connected components of $S_{|\,.\,|^b}(f \circ F; \varepsilon_5, C_7)$ including $F^{-1}(\zeta_\nu)$ is contained in $F^{-1}(V_\nu)$. In fact, it follows from (4.11) that for $z \in \hat{V}_\nu$

$$\begin{aligned} |f \circ F(z)| &< \varepsilon_4 \exp(-\beta C_6) \exp(-\alpha C_6 |z|^b) \\ &\le \varepsilon_4 \exp\left(-\beta C_6 - \alpha C_6 \cdot \frac{|F(z)|^a - \beta}{\alpha}\right) \\ &= \varepsilon_4 \exp(-C_6 |F(z)|^a), \end{aligned}$$

which implies that $F(z) \in S_{|\,.\,|^a}(f; \varepsilon_4, C_6)$. For $z' \in F^{-1}(\zeta_\nu)$, the above inequality holds on every curve through z and z' in $\hat{V}_\nu$. The connectedness of V_ν proves that $z \in F^{-1}(V_\nu)$. It is clear that $\hat{V}_\nu \cap F^{-1}(Z(f_1, \dots, f_m) \setminus \{\zeta_\nu\}) = \emptyset$ for all $\nu \in \mathbb{N}$ by the above argument.

Here we need the following lemma:

Lemma 4.1 (cf. [2, Lemma 3.2]). *Let $f_1, \dots, f_m \in A_p(\mathbb{C}^m)$. Then there exist constants ε, $C > 0$ such that*

$$\sum_{j=1}^{m} |D_u f_j(\zeta_\nu)| \ge \varepsilon \exp(-C p(\zeta_\nu))$$

for all $\nu \in \mathbb{N} \setminus E$ and $u \in S^{2m-1}$ if and only if we have constants ε', $C' > 0$ satisfying

$$|\det Jf(\zeta_\nu)| \ge \varepsilon' \exp(-C' p(\zeta_\nu))$$

for all $\nu \in \mathbb{N}$, where Jf is the Jacobian matrix of $f = (f_1, \dots, f_m)$.

We apply this lemma to obtain ε_6, $C_8 > 0$ such that

$$|\det Jf(\zeta_\nu)| \ge \varepsilon_6 \exp(-C_8 |\zeta_\nu|^a) \tag{4.12}$$

for all $\nu \in \mathbb{N}$. Calculating a sum of the moduli of all $m \times m$ minors of $J(f \circ F)$, we have from (2) of Main Theorem, (4.11) and (4.12)

$$\sum_{\kappa=1}^{\binom{n}{m}} |\triangle_\kappa^{f \circ F}(z)| = |\det Jf(F(z))| \cdot \sum_{\kappa=1}^{\binom{n}{m}} |\triangle_\kappa^{F}(z)|$$

$$
\begin{aligned}
&\geq \varepsilon_6 \exp(-C_8|F(z)|^a) \cdot \varepsilon \exp(-C|z|^b) \\
&\geq (\varepsilon\varepsilon_6 e^{-\beta C_8}) \exp(-(\alpha C_8 + C)|z|^b)
\end{aligned}
$$

for all $z \in \bigcup_{\nu\in\mathbb{N}\setminus E} F^{-1}(\zeta_\nu)$.

Here the proof of [5, Theorem 1] implies the following:

Lemma 4.2. *For $f_1, \dots, f_N \in A_p(\mathbb{C}^n)$, let Z' be a union of connected components of $Z(f_1, \dots, f_N)$ which are k-codimensional manifolds, so that*

$$\sum_{\vartheta=1}^{\binom{N}{k}} \sum_{\kappa=1}^{\binom{n}{k}} |\triangle^f_{\vartheta,\kappa}(z)| \geq \varepsilon \exp(-Cp(z))$$

for all $z \in Z'$, where the sum is taken over all $k \times k$ minors of the Jacobian matrix Jf. If we can choose constants ε'', $C'' > 0$ such that every connected component of $S_p(f; \varepsilon'', C'')$ including a connected component of Z' does not intersect the other connected components of $Z(f_1, \dots, f_N)$, then we have constants $\varepsilon''' < \varepsilon''$, $C''' > C''$ satisfying: Let Y be a connected component of Z' and let V_Y be the connected component of $S_p(f; \varepsilon''', C''')$ including Y. Then there exists a holomorphic retract Φ_Y from V_Y onto Y such that $|z - \Phi_Y(z)| \leq 1$ for all $z \in V_Y$.

By setting $\varepsilon'' = \varepsilon_5$, $C'' = C_7$ and $Z' = \bigcup_{\nu\in\mathbb{N}\setminus E} F^{-1}(\zeta_\nu)$, we can apply this lemma to obtain ε_7, $C_9 > 0$ and a holomorphic retract Φ_ν from $\tilde{V}_\nu$ onto $F^{-1}(\zeta_\nu)$ such that

$$|z - \Phi_\nu(z)| \leq 1 \tag{4.13}$$

for all $\nu \in \mathbb{N} \setminus E$, where $\tilde{V}_\nu$ ($\nu \in \mathbb{N}$) is the union of the connected components of $S_{|\,\cdot\,|^b}(f \circ F; \varepsilon_7, C_9)$ including $F^{-1}(\zeta_\nu)$. It is clear that $\tilde{V}_\nu \cap \tilde{V}_{\nu'} = \emptyset$ for $\nu \neq \nu'$.

For $h \in A_{|\,\cdot\,|^b}(F^{-1}(X))$, it follows from Theorem 3.1 that there exists $\tilde{H} \in A_{|\,\cdot\,|^b}(\mathbb{C}^n)$ such that $\tilde{H}|_{\bigcup_{\nu\in E} F^{-1}(\zeta_\nu)} \equiv h|_{\bigcup_{\nu\in E} F^{-1}(\zeta_\nu)}$. Then we define

$$\tilde{h}(z) = \begin{cases} \Phi_\nu^* h(z), & \text{if } z \in \tilde{V}_\nu \text{ and } \nu \in \mathbb{N} \setminus E, \\ \tilde{H}(z), & \text{if } z \in \tilde{V}_\nu \text{ and } \nu \in E, \\ 0, & \text{if } z \in S_{|\,\cdot\,|^b}(f \circ F; \varepsilon_7, C_9) \setminus \bigcup_{\nu\in\mathbb{N}} \tilde{V}_\nu. \end{cases}$$

Since $|\cdot|^b$ is a weight function, (4.13) implies that there exist A_2, $B_2 > 0$ such that $\tilde{h}(z) \leq A_2 \exp(B_2|z|^b)$ for all $z \in S_{|\,\cdot\,|^b}(f \circ F; \varepsilon_7, C_9)$. Hence it follows from the semilocal interpolation theorem that we obtain $H \in A_{|\,\cdot\,|^b}(\mathbb{C}^n)$ with $H|_{F^{-1}(X)} \equiv h$. Thus $F^{-1}(X)$ is interpolating for $A_{|\,\cdot\,|^b}(\mathbb{C}^n)$.

5. Examples and remarks

The following is an easy example for the main theorem:

EXAMPLE 5.1. Set $X = \{\nu\}_{\nu\in\mathbb{Z}} \subset \mathbb{C}$. By applying Theorem A (or [3, Corollary 3.5]) to $f(z) = \sin 2\pi z \in A_{|\,\cdot\,|}(\mathbb{C})$, we know that V is interpolating for $A_{|\,\cdot\,|}(\mathbb{C})$. Put $F(z_1, \ldots, z_n) = z_1^2 + \cdots + z_n^2$, which satisfies

$$\sum_{j=1}^{n} \left| \frac{\partial F(z)}{\partial z_j} \right| \geq |\operatorname{grad} F(z)| = 2|z|.$$

If $z \in F^{-1}(\nu)$, we have $|z|^2 \geq |\nu|$. In particular, for $\nu \in \mathbb{Z} \setminus \{0\}$, it follows that

$$\sum_{j=1}^{n} \left| \frac{\partial F(z)}{\partial z_j} \right| \geq 2\sqrt{|\nu|} \geq 2.$$

Hence the main theorem implies that $F^{-1}(X)$ is interpolating for $A_{|\,\cdot\,|^b}(\mathbb{C}^n)$ for all $b \geq 2$. (In this case, $E = \{0\}$.)

In the case where $n = m = 1$, we can improve the main theorem as follows:

Corollary 5.2. *Let $X = \{\zeta_\nu\}_{\nu\in\mathbb{N}}$ be a discrete variety in $\mathbb{C}$ and let $F \in \mathbb{C}[z]$. Put $d = \deg F$. For $a > 0$, we assume that X is interpolating for $A_{|\,\cdot\,|^a}(\mathbb{C})$. Then $F^{-1}(X)$ is interpolating for $A_{|\,\cdot\,|^b}(\mathbb{C})$ for every $b \geq ad$.*

Finally, we remark that the term '$b \geq ad$' in the main theorem is sharp in the sense of the following open problem:

Open Problem ([5, Problem 1]). Let q be another weight function on $\mathbb{C}^n$ satisfying $q \geq p$ everywhere. Assume that an analytic subset X of $\mathbb{C}^n$ is interpolating for $A_p(\mathbb{C}^n)$. Then is V interpolating for $A_q(\mathbb{C}^n)$?

We prove this remark by giving an example for which $F^{-1}(X)$ is not interpolating for $A_{|\,\cdot\,|^b}(\mathbb{C}^n)$ for $b < ad$. Let $X = \{\zeta_\nu\}_{\nu\in\mathbb{N}}$ be a discrete variety in $\mathbb{C}$. Then Nevanlinna's counting function is defined as follows: $n(r, \zeta, X) = \sharp\{\nu \in \mathbb{N} : |\zeta_\nu - \zeta| \leq r\}$ and

$$N(r, \zeta, X) = \int_0^r \frac{n(t, \zeta, X) - n(0, \zeta, X)}{t} dt + n(0, \zeta, X) \log r.$$

EXAMPLE 5.3. Assume that $n = m = 1$. Put $X = \{\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{C}$. As in Example 5.1, it follows that X is interpolating for $A_{|\,\cdot\,|}(\mathbb{C})$. Set $F(z) = z^4$, so $\deg F = 4$. Then we have

$$F^{-1}(X) = \left\{ \sqrt[4]{\nu} \cdot \exp\left(\frac{l\pi i}{2}\right) \right\}_{\nu\in\mathbb{N};\, l=0,1,2,3}.$$

Corollary 5.2 implies that $F^{-1}(X)$ is interpolating for $A_{|\cdot|^b}(\mathbb{C})$ for every $b \geq 4$.

Here we claim that $F^{-1}(X)$ is not interpolating for $A_{|\cdot|^b}(\mathbb{C})$ for any $b < 4$. In fact, we have $n(r, 0, F^{-1}(X)) = 4\nu$ when $\sqrt[4]{\nu} \leq r < \sqrt[4]{\nu + 1}$, so $n(r, 0, F^{-1}(X)) = 4[r^4]$, where $[x] = \max\{y \in \mathbb{Z} : y \leq x\}$. Since $n(s, 0, F^{-1}(X)) = 0$ for all $s \in [0, 1)$ and $[t^4] \geq t^4 - 1$ for all $t \in \mathbb{R}$, we obtain

$$\begin{aligned} N(r, 0, F^{-1}(X)) &= \int_0^r \frac{4[t^4]}{t} dt \\ &\geq \int_1^r \frac{4(t^4 - 1)}{t} dt \\ &= r^4 - 4\log r - 1. \end{aligned}$$

Hence for every $b < 4$ there do not exist two constants A, $B > 0$ such that

$$N(r, 0, F^{-1}(X)) \leq Ar^b + B$$

for all $r \geq 0$. Then it follows from [3, Corollary 4.8] that $F^{-1}(X)$ is not interpolating for $A_{|\cdot|^b}(\mathbb{C})$ for any $b < 4$.

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Graduate School of Mathematical Science
University of Tokyo
3-8-1 Komaba, Meguro
Tokyo 153-8914, Japan
e-mail: oh-uchi@ms.u-tokyo.ac.jp